

9706076
UTHEP-316
June, 1997

ADE Confining Phase Superpotentials

Seiji Terashima and Sung-Kil Yang

Institute of Physics, University of Tsukuba, Ibaraki 305, Japan

Abstract

We obtain a low-energy effective superpotential for a phase with a single confined photon in $N = 1$ gauge theory with an adjoint matter with ADE gauge groups. The expectation values of gauge invariants built out of the adjoint field parametrize the singularities of moduli space of the Coulomb phase. The result can be used to derive the $N = 2$ curve in the form of a foliation over \mathbf{CP}^1 . Our $N = 1$ theory exhibits non-trivial fixed points which naturally inherit the properties of the ADE classification of $N = 2$ superconformal field theories in four dimensions. We also discuss how to include matter hypermultiplets toward deriving the Riemann surface which describes $N = 2$ QCD with exceptional gauge groups.

1 Introduction

Recently $N = 2$ supersymmetry has played a profound role in understanding strong-coupling dynamics of gauge and string theories in various dimensions [1]. In four dimensions the vacuum structure of the $N = 2$ Coulomb phase is described in terms of the Riemann surface [2]. This geometry of $N = 2$ Yang-Mills theory is called Seiberg-Witten (SW) geometry.

An interesting approach to the issue of SW geometry is based on $N = 1$ supersymmetry. When $N = 2$ theory is perturbed by a tree-level superpotential explicitly breaking $N = 2$ to $N = 1$ supersymmetry it is observed that only the singularities of moduli space where monopoles or dyons become massless remain as the $N = 1$ vacua [2]. Thus, studying the low-energy properties of $N = 1$ Yang-Mills theory with an adjoint matter field with a tree-level superpotential chosen properly one may derive the singular loci of $N = 2$ moduli space [3],[4],[5]. For this purpose, using the “integrating-in” technique [6],[3], Elitzur et al. have developed a method of $N = 1$ confining phase superpotential by focusing on a phase with a single confined photon [7]. This approach has now been extended for supersymmetric Yang-Mills theory as well as QCD with all classical gauge groups [8],[9],[10],[11]. In the case of exceptional gauge groups only G_2 gauge theory has been analyzed so far [12].

In this article we wish to show that the method described above applies in a unified way in determining the singularity structure of moduli space of the Coulomb phase in supersymmetric gauge theories with ADE gauge groups. Not only the classical case of A_r, D_r groups but the exceptional case of E_6, E_7, E_8 groups can be treated on an equal footing since our discussion is based on the fundamental properties of the root system of the simply-laced Lie algebras.

For exceptional gauge groups there has accumulated considerable evidence that SW geometry is not realized by hyperelliptic curves [13],[12],[14],[15],[16]. In fact we will see that the Riemann surface described as a foliation over \mathbf{CP}^1 satisfies the singularity conditions we obtain from the $N = 1$ confining phase superpotential. This Riemann surface is not of hyperelliptic type for exceptional gauge groups.

In sect.2 we derive the low-energy effective superpotential which is used to determine

the singularities of moduli space in $N = 1$ theory, which in turn enables us to construct $N = 2$ SW geometry for ADE gauge groups. In sect.3 the ADE series of $N = 1$ superconformal field theories realized at particular vacuum in the Coulomb phase is discussed. In sect.4 we study how the results in sect.2 obtained for $N = 1$ pure Yang-Mills theory with an adjoint matter is extended so as to include chiral matter multiplets. Finally in sect.5 we draw our conclusions.

2 $N=1$ theory with an adjoint matter

Let us consider $N = 1$ supersymmetric gauge theory with an adjoint matter Φ . We assume that the gauge group G is simple and simply-laced, namely, G is of ADE type. Our purpose in this section is to show that, under appropriate ansatz, the low-energy effective superpotential for the Coulomb phase is obtained in a unified way for all ADE gauge groups just by using the fundamental properties of the root system Δ . Our notation for the root system is as follows. The simple roots of G are denoted as α_i where $1 \leq i \leq r$ with r being the rank of G . Any root is decomposed as $\alpha = \sum_{i=1}^r a^i \alpha_i$. The component indices are lowered by $a_i = \sum_{j=1}^r A_{ij} a^j$ where A_{ij} is the ADE Cartan matrix. The inner product of two roots α, β are then defined by

$$\alpha \cdot \beta = \sum_{i=1}^r a^i b_i = \sum_{i,j=1}^r a^i A_{ij} b^j, \quad (1)$$

where $\beta = \sum_{i=1}^r b^i \alpha_i$. For ADE all roots have the equal norm and we normalize $\alpha^2 = 2$.

In our $N = 1$ theory we take a tree-level superpotential

$$W = \sum_{k=1}^r g_k u_k(\Phi), \quad (2)$$

where u_k is the k -th Casimir of G constructed from Φ and g_k are coupling constants. The mass dimension of u_k is $e_k + 1$ with e_k being the k -th exponent of G . When $g_k = 0$ Φ is considered as the chiral field in the $N = 2$ vector multiplet and we have $N = 2$ ADE supersymmetric gauge theory.

We first make a classical analysis of the theory with the superpotential (2). The classical vacua are determined by the equation of motion $\frac{\partial W}{\partial \Phi} = 0$ and the D -term equation. Due to the D -term equation, we can restrict Φ to take the values in the Cartan subalgebra

by the gauge rotation. We denote the vector in the Cartan subalgebra corresponding to the classical value of Φ as $a = \sum_{i=1}^r a^i \alpha_i$. Then the superpotential becomes

$$W(a) = \sum_{k=1}^r g_k u_k(a), \quad (3)$$

and the equation of motion reads

$$\frac{\partial W(a)}{\partial a^i} = \sum_{k=1}^r g_k \frac{\partial u_k(a)}{\partial a^i} = 0. \quad (4)$$

For $g_k \neq 0$ we must have

$$J(a) \equiv \det \left(\frac{\partial u_j(a)}{\partial a^i} \right) = 0. \quad (5)$$

According to [17] it follows that

$$J(a) = c_1 \prod_{\alpha \in \Delta^+} a \cdot \alpha, \quad (6)$$

where Δ^+ is a set of positive roots and c_1 is a certain constant.

The condition $J(a) = 0$ means that the vector a hits a wall of the Weyl chamber and there occurs enhanced gauge symmetry. Suppose that the vector a is orthogonal to a root, say, α_1

$$a \cdot \alpha_1 = 0, \quad (7)$$

where α_1 may be taken to be a simple root. In this case we have the unbroken gauge group $SU(2) \times U(1)^{r-1}$ where the $SU(2)$ factor is spanned by $\{\alpha_1 \cdot H, E_{\alpha_1}, E_{-\alpha_1}\}$ in the Cartan-Weyl basis. If some other factors of J vanish besides $a \cdot \alpha_1$ the gauge group is further enhanced from $SU(2)$. Since $SU(2) \times U(1)^{r-1}$ is the most generic unbroken gauge group we shall restrict ourselves to this case in what follows.

We remark here that there is the case in which the $SU(2) \times U(1)^{r-1}$ vacuum is not generic. As a simple, but instructive example consider $SU(4)$ theory. Casimirs are taken to be

$$\begin{aligned} u_1 &= \frac{1}{2} \text{Tr } \Phi^2, \\ u_2 &= \frac{1}{3} \text{Tr } \Phi^3, \\ u_3 &= \frac{1}{4} \text{Tr } \Phi^4 - \alpha \left(\frac{1}{2} \text{Tr } \Phi^2 \right)^2, \end{aligned} \quad (8)$$

where α is an arbitrary constant. If we set $\alpha = 1/2$ it is observed that the $SU(2) \times U(1)^2$ vacuum exists only for the special values of coupling constants, $(g_2/g_3)^2 = g_1/g_3$. Thus, for $\alpha = 1/2$, the $SU(2) \times U(1)^2$ vacuum is not generic though it does so for $\alpha \neq 1/2$. This points out that we have to choose the appropriate basis for Casimirs when writing down (2) to have the $SU(2) \times U(1)^{r-1}$ vacuum generically [8].

Now we assume that there is no mixing between the $SU(2) \times U(1)^{r-1}$ vacuum and other vacua with different unbroken gauge groups. According to the arguments of [18], we should not consider the broken gauge group instantons. We thus expect that there is only perturbative effect in the energy scale above the scale Λ_{YM} of the low-energy effective $N = 1$ supersymmetric $SU(2)$ Yang-Mills theory.

2.1 Higgs mass

Our next task is to evaluate the Higgs scale associated with the spontaneous breaking of the gauge group G to $SU(2) \times U(1)^{r-1}$. For this purpose we decompose the adjoint representation of G to irreducible representations of $SU(2)$. We fix the $SU(2)$ direction by taking a simple root α_1 . It is clear that the spin j of every representation obtained in this decomposition satisfies $j \leq 1$ since all roots have the same norm and the $SU(2)$ raising (or lowering) operator shifts a root α to $\alpha + \alpha_1$ (or $\alpha - \alpha_1$). The fact that there is no degeneration of roots indicates that the $j = 1$ multiplet has the roots $(\alpha_1, 0, -\alpha_1)$ corresponding to the unbroken $SU(2)$ generators. The roots orthogonal to α_1 represent the $j = 0$ multiplets. The $j = 1/2$ multiplets have the roots α obeying $\alpha \cdot \alpha_1 = \pm 1$. Let us define a set of these roots by $\Delta_d = \{\alpha | \alpha \in \Delta, \alpha \cdot \alpha_1 = \pm 1\}$. For each root $\alpha \in \Delta_d$ there appears a massive gauge boson. These massive bosons pair up in $SU(2)$ doublets with weights $(\alpha, \alpha \pm \alpha_1)$ which indeed have the same mass $|a \cdot \alpha| = |a \cdot (\alpha \pm \alpha_1)|$ since $a \cdot \alpha_1 = 0$.

We now integrate out the fields that become massive by the Higgs mechanism. The massless $U(1)^{r-1}$ degrees of freedom are decoupled. The resulting theory characterized by the scale Λ_H is $N = 1$ $SU(2)$ theory with an adjoint chiral multiplet. The Higgs scale

Λ_H is related to the high-energy scale Λ through the scale matching relation

$$\Lambda^{2h} = \Lambda_H^{2\cdot 2} \left(\prod_{\beta \in \Delta_d, \beta > 0} a \cdot \beta \right)^\ell, \quad (9)$$

where $2h = 4 + \ell n_d/2$, n_d is the number of elements in Δ_d and h stands for the dual Coxeter number of G ; $h = r+1, 2r-2, 12, 18, 30$ for $G = A_r, D_r, E_6, E_7, E_8$ respectively. The reason for $\beta > 0$ in (9) is that weights $(\beta, \beta \pm \alpha_1)$ of an $SU(2)$ doublet are either both positive or both negative since α_1 is the simple root, and gauge bosons associated with $\beta < 0$ and $\beta > 0$ have the same contribution to the relation (9).

To fix ℓ we calculate n_d by evaluating the quadratic Casimir C_2 of the adjoint representation in the following way. Taking hermitian generators we express C_2 in terms of the structure constants f_{abc} through $\sum_{a,b} f_{abc} f_{abc'} = -C_2 \delta_{cc'}$. From the commutation relation $[\alpha_1 \cdot H, E_\alpha] = (\alpha_1 \cdot \alpha) E_\alpha$ one can check

$$C_2 = \frac{1}{2} \sum_{\alpha \in \Delta} (\alpha_1 \cdot \alpha)^2 = \frac{1}{2} \left(\sum_{\alpha \in \Delta_d} (\alpha_1 \cdot \alpha)^2 + 2(\alpha_1 \cdot \alpha_1)^2 \right) = \frac{1}{2} (n_d + 8). \quad (10)$$

On the other hand, the dual Coxeter number h is given by $h = C_2/\theta^2$ with θ being the highest root. We thus find

$$n_d = 4(h-2) \quad (11)$$

and (9) becomes

$$\Lambda^{2h} = \Lambda_H^{2\cdot 2} \prod_{\beta \in \Delta_d, \beta > 0} a \cdot \beta. \quad (12)$$

2.2 Adjoint mass

After integrating out the massive fields due to the Higgs mechanism we are left with $N = 1$ $SU(2)$ theory with the massive adjoint. In order to evaluate the mass of the adjoint chiral multiplet Φ we need to clarify some properties of Casimirs. Let σ_β be an element of the Weyl group of G specified by a root $\beta = \sum_{i=1}^r b^i \alpha_i$. The Weyl transformation of a root α is given by

$$\sigma_\beta(\alpha) = \alpha - (\alpha \cdot \beta) \beta. \quad (13)$$

When σ_β acts on the Higgs v.e.v. vector $a = \sum_{i=1}^r a^i \alpha_i$ we have

$$a'^i = \sum_{j=1}^r S_\beta^i{}_j a^j, \quad S_\beta^i{}_j \equiv \delta^i_j - b^i b_j, \quad (14)$$

where $\sigma_\beta(a) = \sum_{i=1}^r a^{i\beta} \alpha_i$. Since the Casimirs $u_k(a)$ are Weyl invariants it is obvious to see

$$\frac{\partial}{\partial a^i} u_k(a) = \frac{\partial}{\partial a^i} u_k(a') = \sum_{j=1}^r S_{\beta}{}^j{}_i \left(\frac{\partial}{\partial a^j} u_k(a) \right) \Big|_{a \rightarrow a'}. \quad (15)$$

Let \bar{a} be a particular v.e.v. which is fixed under the action of σ_β , then we find the identity

$$\sum_{j=1}^r \left(\delta_i^j - S_{\beta}{}^j{}_i \right) \frac{\partial}{\partial a^j} u_k(a) \Big|_{a=\bar{a}} = 0 \quad (16)$$

for all i , and thus

$$\sum_{j=1}^r b_j \frac{\partial}{\partial a^j} u_k(a) \Big|_{a=\bar{a}} = 0. \quad (17)$$

This implies that for any v.e.v. vector a and root β we can write down

$$\sum_{j=1}^r b^j \frac{\partial}{\partial a^j} u_k(a) = (a \cdot \beta) u_k^\beta(a), \quad (18)$$

where $u_k^\beta(a)$ is some polynomial of a^i . If we set $\beta = \alpha_i$, a simple root, we obtain a useful formula

$$\frac{\partial}{\partial a^i} u_k(a) = a_i u_k^{\alpha_i}(a). \quad (19)$$

As an immediate application of the above results, for instance, we point out that (6) is derived from (18) and the fact that the mass dimension of $J(a)$ is given by

$$\sum_{k=1}^r e_k = \frac{1}{2} (\dim G - r), \quad (20)$$

where e_k is the k -th exponent of G .

Let us further discuss the properties of $u_k^{\alpha_j}(a)$. Define D_{mn} as

$$D_{mn} \equiv (-1)^{n+m} \det \left(\frac{\partial u_{\tilde{j}}(a)}{\partial a^{\tilde{i}}} \right), \quad 1 \leq m, n \leq r, \quad (21)$$

where $1 \leq \tilde{i}, \tilde{j} \leq r$ with $\tilde{i} \neq m, \tilde{j} \neq n$, then D_{1n} is a homogeneous polynomial of a^i with the mass dimension $\sum_{k=1}^r e_k - e_n$. We also denote Δ_e as a set of positive roots where α_1 and $SU(2)$ doublet roots α with $\alpha + \alpha_1 \notin \Delta^+$ are excluded. If we set $a_1 = 0$ and $a \cdot \beta = 0$ where β is any root in Δ_e we see $D_{1n} = 0$ from the identity (18). Consequently we can expand

$$D_{1n} = h_n(a) \prod_{\beta \in \Delta_e} (a \cdot \beta) + a_1 f_n(a), \quad (22)$$

where $h_n(a)$, $f_n(a)$ are polynomials of a_i . In particular

$$D_{1r} = c_2 \prod_{\beta \in \Delta_e} (a \cdot \beta) + a_1 f_r(a), \quad (23)$$

where c_2 is a constant. Notice that the first term on the rhs has the correct mass dimension since the number of roots in Δ_e reads

$$\frac{1}{2} (\dim G - r) - 1 - \frac{n_d}{4} = \sum_{k=1}^r e_k - (h - 1), \quad (24)$$

where we have used (11) and $e_r = h - 1$.

We are now ready to evaluate the mass of Φ in intermediate $SU(2)$ theory. The fluctuation of $W(a)$ around the classical vacuum yields the adjoint mass. To find the mass relevant for the scale matching we should only consider the components of Φ which are coupled to the unbroken $SU(2)$. The mass M_Φ of these components is then given by

$$2M_\Phi = \left. \frac{\partial^2}{(\partial a^1)^2} W(a) \right| = \left. \frac{\partial}{\partial a^1} (a_1 W_1) \right| = \left. \left(a_1 \frac{\partial}{\partial a^1} W_1 + 2W_1 \right) \right| = 2W_1|, \quad (25)$$

where $W_1 = (\sum_{k=1}^r g_k u_k^{\alpha_1})(a)$ and a^i are understood as solutions of the equation of motion (4).

To proceed further it is convenient to rewrite the equation motion (4) and the vacuum condition (7) with the simple root α_1 as follows:

$$\begin{aligned} g_1 : g_2 : \cdots : g_r &= D_{11} : D_{12} : \cdots : D_{1r}, \\ a_1 &= 0. \end{aligned} \quad (26)$$

The solutions of these equations are expressed as functions of the ratio g_i/g_r . Then we notice that $J(a)$ defined in (5) turns out to be

$$J = \sum_{k=1}^r \frac{\partial u_k}{\partial a^1} D_{1k} = \frac{D_{1r}}{g_r} \sum_{k=1}^r g_k \frac{\partial u_k}{\partial a^1} = D_{1r} a_1 \frac{W_1}{g_r}. \quad (27)$$

Combining (6) and (23) we obtain

$$M_\Phi^2 = (W_1|)^2 = \left(\frac{c_1}{c_2} \right)^2 g_r^2 \prod_{\beta \in \Delta_d, \beta > 0} a \cdot \beta. \quad (28)$$

Upon integrating out the massive adjoint we relate the scale Λ_H with the scale Λ_{YM} of the low-energy $N = 1$ $SU(2)$ Yang-Mills theory by

$$\Lambda_H^{2:2} = \Lambda_{YM}^{3:2} / M_\Phi^2. \quad (29)$$

We finally find from this and (9), (28) that the scale matching relation becomes

$$\Lambda_{YM}^{3:2} = g_r^2 \Lambda^{2h}, \quad (30)$$

where the top Casimir u_r has been rescaled so that we can set $c_1/c_2 = 1$.

Following the previous discussions and the perturbative nonrenormalization theorem for the superpotential, we derive the low-energy effective superpotential

$$W_L = W_{cl}(g) \pm 2\Lambda_{YM}^3 = W_{cl}(g) \pm 2g_r \Lambda^h, \quad (31)$$

where the term $\pm 2\Lambda_L^3$ has appeared as a result of the gaugino condensation in low-energy $SU(2)$ theory and $W_{cl}(g)$ is the tree-level superpotential evaluated at the classical values $a^i(g)$. We will assume that (31) is the exact effective superpotential valid for all values of parameters.

2.3 Determination of singularities and $N = 2$ curves

The vacuum expectation values of gauge invariants are obtained from W_L

$$\langle u_k \rangle = \frac{\partial W_L}{\partial g_k} = u_k^{cl}(g) \pm 2\Lambda^h \delta_{k,r}. \quad (32)$$

We now wish to show that the expectation values (32) parametrize the singularities of algebraic curves. For this let us introduce

$$P_{\mathcal{R}}(x, u_k^{cl}) = \det(x - \Phi_{\mathcal{R}}) \quad (33)$$

which is the characteristic polynomial in x of order $\dim \mathcal{R}$ where \mathcal{R} is an irreducible representation of G . Here $\Phi_{\mathcal{R}}$ is a representation matrix of \mathcal{R} and u_k^{cl} are Casimirs built out of $\Phi_{\mathcal{R}}$. The eigenvalues of $\Phi_{\mathcal{R}}$ are given in terms of the weights λ_i of the representation \mathcal{R} . Diagonalizing $\Phi_{\mathcal{R}}$ we may express (33) as

$$P_{\mathcal{R}}(x, a) = \prod_{i=1}^{\dim \mathcal{R}} (x - a \cdot \lambda_i), \quad (34)$$

where a is a Higgs v.e.v. vector, the discriminant of which takes the form

$$\Delta_{\mathcal{R}} = \left(\prod_{i \neq j} a \cdot (\lambda_i - \lambda_j) \right)^2. \quad (35)$$

It is seen that, for a which is a solution to (4), we have $\Delta_{\mathcal{R}} = 0$, that is

$$P_{\mathcal{R}}(x, u_k^{cl}(a)) = \partial_x P_{\mathcal{R}}(x, u_k^{cl}(a)) = 0 \quad (36)$$

for any representation. The solutions of the classical equation of motion thus give rise to the singularities of the level manifold $P_{\mathcal{R}}(x, u_k^{cl}) = 0$.

In order to include the quantum effect what we should do is to modify the top Casimir u_r term so that the gluino condensation in (32) is properly taken into account. We are then led to take a curve

$$\tilde{P}_{\mathcal{R}}(x, z, u_k) \equiv P_{\mathcal{R}} \left(x, u_k + \delta_{k,r} \left(z + \frac{\mu}{z} \right) \right) = 0, \quad (37)$$

where $\mu = \Lambda^{2h}$ and an additional complex variable z has been introduced. Let us check the degeneracy of the curve at the expectation values (32), which means to check if the following three equations hold

$$\tilde{P}_{\mathcal{R}}(x, z, \langle u_k \rangle) = 0, \quad (38)$$

$$\partial_x \tilde{P}_{\mathcal{R}}(x, z, \langle u_k \rangle) = 0, \quad (39)$$

$$\partial_z \tilde{P}_{\mathcal{R}}(x, z, \langle u_k \rangle) = \left(1 - \frac{\mu}{z^2} \right) \partial_{u_r} \tilde{P}_{\mathcal{R}}(x, z, \langle u_k \rangle) = 0. \quad (40)$$

The last equation (40) has an obvious solution $z = \mp\sqrt{\mu}$. Substituting this into the first two equations we see that the singularity conditions reduce to the classical ones (36)

$$\tilde{P}_{\mathcal{R}}(x, \mp\sqrt{\mu}, \langle u_k \rangle) = P_{\mathcal{R}}(x, \langle u_k \rangle \mp \delta_{k,r} 2\sqrt{\mu}) = P_{\mathcal{R}}(x, u_k^{cl}) = 0, \quad (41)$$

$$\partial_x \tilde{P}_{\mathcal{R}}(x, \mp\sqrt{\mu}, \langle u_k \rangle) = \partial_x P_{\mathcal{R}}(x, \langle u_k \rangle \mp \delta_{k,r} 2\sqrt{\mu}) = \partial_x P_{\mathcal{R}}(x, u_k^{cl}) = 0. \quad (42)$$

Thus we have shown that (32) parametrize the singularities of the Riemann surface described by (37) irrespective of the representation \mathcal{R} .

Let us take the $N = 2$ limit by letting all $g_i \rightarrow 0$ with the ratio g_i/g_r fixed, then (37) is the curve describing the Coulomb phase of $N = 2$ supersymmetric Yang-Mills theory

with ADE gauge groups. Indeed the curve (37) in this particular form of foliation agrees with the one obtained systematically in [13] in view of integrable systems [19],[20],[21]. For E_6 and E_7 see [14],[15].

Finally we remark that there is a possibility of (40) having another solutions besides $z = \mp\sqrt{\mu}$. If we take the fundamental representation such solutions are absent for $G = A_r$, and for $G = D_r$ there is a solution with vanishing degree r Casimir (i.e. Pfaffian), but it is known that this is an apparent singularity [22]. For E_r gauge groups there could exist additional solutions. We expect that these singularities are apparent and do not represent physical massless solitons.

3 Superconformal field theories

We will discuss non-trivial fixed points in our $N = 1$ theory characterized by the microscopic superpotential (2). To find critical points we rely on the recent construction of new $N = 2$ superconformal field theories realized at particular points in the moduli space of the Coulomb phase [23],[24],[25],[26]. At these $N = 2$ critical points mutually non-local massless dyons coexist. Thus the critical points lie on the singularities in the moduli space which are parametrized by the $N = 1$ expectation values (32) as was shown in the previous section. This enables us to adjust the microscopic parameters in $N = 1$ theory to the values of $N = 2$ non-trivial fixed points. Doing so in $N = 2$ $SU(3)$ Yang-Mills theory Argyres and Douglas found non-trivial $N = 1$ fixed points [23]. We now show that this class of $N = 1$ fixed points exists in all ADE $N = 1$ theories in general. See [8] for discussions on AD theories.

Let us start with rederiving $N = 2$ critical behavior based on the curve (37). An advantage of using the curve (37) is that one can identify higher critical points and determine the critical exponents independently of the details of the curve.

If we set $z = \mp\sqrt{\mu}$ the condition for higher critical points is

$$P_{\mathcal{R}}(x, u_k^{cl}) = \partial_x^n P_{\mathcal{R}}(x, u_k^{cl}) = 0 \quad (43)$$

with $n > 2$. Hence there exist higher critical points at $u_k = u_k^{sing} \pm 2\Lambda^h \delta_{k,r}$ where u_k^{sing} are the classical values of u_k for which the gauge group H with rank larger than one is

left unbroken. The highest critical point corresponding to the unbroken G is located at $u_k = \pm 2\Lambda^h \delta_{k,r}$.

Near the highest critical point the curve (37) behaves as

$$u_r + z + \frac{\mu}{z} = c x^h + \delta u_k x^j, \quad (44)$$

where the second term on the rhs with $j = h - (e_k + 1)$ represents a small perturbation around the criticality at $\delta u_k = 0$. A constant c is irrelevant and will be set to $c = 1$. Let $u_r = \pm 2\Lambda^h$, $x = \delta u_k^{1/(h-j)} s$ and $z \pm \Lambda^h = \rho$, then (44) becomes

$$\rho \simeq \delta u_k^{\frac{h}{2(h-j)}} (\mp \Lambda^h)^{\frac{1}{2}} (s^h + s^j)^{\frac{1}{2}}. \quad (45)$$

We now apply the technique of [25] to verify the scaling behavior of the period integral of the Seiberg-Witten differential λ_{SW} . For the curve (37) it is known that $\lambda_{SW} = x dz/z$. Near the critical value $z = \mp \sqrt{\mu}$ we evaluate

$$\begin{aligned} \oint \lambda_{SW} &= \oint x \frac{dz}{z} \simeq \oint x d\rho \\ &\simeq \delta u_k^{\frac{h+2}{2(h-j)}} \oint ds \frac{hs^h + js^j}{(s^h + s^j)^{1/2}}. \end{aligned} \quad (46)$$

Since the period has the mass dimension one we read off critical exponents

$$\frac{2(e_k + 1)}{h + 2}, \quad k = 1, 2, \dots, r \quad (47)$$

in agreement with the results obtained earlier for $N = 2$ ADE Yang-Mills theories [25],[26].

When our $N = 1$ theory is viewed as $N = 2$ theory perturbed by the tree-level superpotential (2) we understand that the mass gap in $N = 1$ theory arises from the dyon condensation [2]. Let us show that the dyon condensate vanishes as we approach the $N = 2$ highest critical point under $N = 1$ perturbation. The $SU(2) \times U(1)^{r-1}$ vacuum in $N = 1$ theory corresponds to the $N = 2$ vacuum where a single monopole or dyon becomes massless. The low-energy effective superpotential takes the form

$$W_m = \sqrt{2} A M \tilde{M} + \sum_{k=1}^r g_k U_k, \quad (48)$$

where A is the $N = 1$ chiral superfield in the $N = 2$ $U(1)$ vector multiplet, M, \tilde{M} are the $N = 1$ chiral superfields of an $N = 2$ dyon hypermultiplet and U_k represent the

superfields corresponding to Casimirs $u_k(\Phi)$. We will use lower-case letters to denote the lowest components of the corresponding upper-case superfields. Note that $\langle a \rangle = 0$ in the vacuum with a massless soliton.

The equation of motion $dW_m = 0$ is given by

$$-\frac{g_k}{\sqrt{2}} = \frac{\partial A}{\partial U_k} M \tilde{M}, \quad 1 \leq k \leq r \quad (49)$$

and $AM = A\tilde{M} = 0$, from which we have

$$\frac{g_k}{g_r} = \frac{\partial a / \partial u_k}{\partial a / \partial u_r}, \quad 1 \leq k \leq r-1, \quad (50)$$

when $\langle a \rangle = 0$. The vicinity of $N = 2$ highest criticality may be parametrized by

$$\langle u_k \rangle = \pm 2\Lambda^h \delta_{k,r} + c_k \epsilon^{e_k+1}, \quad c_k = \text{constant}, \quad (51)$$

where ϵ is an overall mass scale. From (46) one obtains

$$\frac{\partial a}{\partial u_k} \simeq \epsilon^{\frac{h}{2}-e_k}, \quad 1 \leq k \leq r, \quad (52)$$

so that

$$\frac{g_k}{g_r} \simeq \epsilon^{h-e_k-1} \longrightarrow 0, \quad 1 \leq k \leq r-1 \quad (53)$$

as $\epsilon \rightarrow 0$. The scaling behavior of dyon condensate is easily derived from (49)

$$\langle m \rangle = \left(-\frac{g_r}{\sqrt{2} \partial a / \partial u_r} \right)^{1/2} \simeq \sqrt{g_r} \epsilon^{(h-2)/4} \longrightarrow 0. \quad (54)$$

Therefore the gap in the $N = 1$ confining phase vanishes. We thus find that $N = 1$ ADE gauge theory with an adjoint matter with a tree-level superpotential

$$W_{\text{crit}} = g_r u_r(\Phi) \quad (55)$$

exhibits non-trivial fixed points. The higher-order polynomial $u_r(\Phi)$ is a dangerously irrelevant operator which is irrelevant at the UV gaussian fixed point, but affects the long-distance behavior significantly [27].

4 Chiral matter multiplets

In this section we consider $N = 1$ gauge theory with N_f flavors of chiral matter multiplets Q^i, \tilde{Q}_j ($1 \leq i, j \leq N_f$) in addition to the adjoint matter Φ . Here Q belongs to an irreducible representation \mathcal{R} of the gauge group G with the dimension d_R and \tilde{Q} belongs to the conjugate representation of \mathcal{R} . We take a tree-level superpotential

$$W = \sum_{k=1}^r g_k u_k(\Phi) + \sum_{l=0}^q \text{Tr}_{N_f} \gamma_l \tilde{Q} \Phi_R^l Q, \quad (56)$$

where Φ_R is a $d_R \times d_R$ matrix representation of Φ in \mathcal{R} and $(\gamma_l)_{ij}$, $1 \leq i, j \leq N_f$, are the coupling constants and q should be restricted so that $\tilde{Q} \Phi_R^l Q$ is irreducible in the sense that it cannot be factored into gauge invariants. If we set $(\gamma_0)_j^i = m_j^i$ with $[m, m^\dagger] = 0$, $(\gamma_1)_j^i = \sqrt{2} \delta_j^i$, $(\gamma_l)_j^i = 0$ for $l > 1$ and all $g_i = 0$, (56) reduces to the superpotential in $N = 2$ supersymmetric Yang-Mills theory with massive N_f hypermultiplets.

Let us focus on the classical vacua of the Coulomb phase with $Q = \tilde{Q} = 0$ and an unbroken $SU(2) \times U(1)^{r-1}$ gauge group symmetry. The vacuum condition for Φ is given by (26) and the classical vacuum takes the form as in the Yang-Mills case

$$\Phi_R = \text{diag}(a \cdot \lambda_1, a \cdot \lambda_2, \dots, a \cdot \lambda_{d_R}), \quad (57)$$

where λ_i are the weights of the representation \mathcal{R} . In this vacuum, we will evaluate semiclassically the low-energy effective superpotential in the tree-level parameter region where the Higgs mechanism occurs at very high energies and the adjoint matter field Φ is quite heavy. Then the massive particles are integrated out and we get low-energy $SU(2)$ theory with flavors.

This integrating-out process results in the scale matching relation which is essentially the same as the the Yang-Mills case (30) except that we here have to take into account flavor loops. The one instanton factor in high-energy theory is given by $\Lambda^{2h-l(\mathcal{R})N_f}$. Here the index $l(\mathcal{R})$ of the representation \mathcal{R} is defined by $l(\mathcal{R})\delta_{ab} = \text{Tr}(T_a T_b)$ where T_a is the representation matrix of \mathcal{R} with root vectors normalized as $\alpha^2 = 2$. The index is always an integer [28]. The scale matching relation becomes

$$\Lambda_L^{3 \cdot 2 - l(\mathcal{R})N_f} = g_r^2 \Lambda^{2h - l(\mathcal{R})N_f}, \quad (58)$$

where Λ_L is the scale of low-energy $SU(2)$ theory with massive flavors.

To consider the superpotential for low-energy $SU(2)$ theory with N_f flavors we decompose the matter representation \mathcal{R} of G in terms of the $SU(2)$ subgroup. We have

$$\mathcal{R} = \bigoplus_{s=1}^{n_{\mathcal{R}}} \mathcal{R}_{SU(2)}^s \oplus \text{singlets}, \quad (59)$$

where $\mathcal{R}_{SU(2)}^s$ stands for a non-singlet $SU(2)$ representation. Accordingly Q^i is decomposed into $SU(2)$ singlets and \mathbf{Q}_s^i ($1 \leq i \leq N_f$, $1 \leq s \leq n_{\mathcal{R}}$) in an $SU(2)$ representation $\mathcal{R}_{SU(2)}^s$. \tilde{Q}_i is decomposed in a similar manner. The singlet components are decoupled in low-energy $SU(2)$ theory.

The semiclassical superpotential for $SU(2)$ theory with N_f flavors is now given by

$$W = \sum_{k=1}^r g_k u_k^{cl} + \sum_{s=1}^{n_{\mathcal{R}}} \sum_{l=0}^q (a \cdot \lambda_{\mathcal{R}_s})^l \text{Tr}_{N_f} \gamma_l \tilde{\mathbf{Q}}_s \mathbf{Q}_s, \quad (60)$$

where $\lambda_{\mathcal{R}_s}$ is a weight of \mathcal{R} which branches to the weights in $\mathcal{R}_{SU(2)}^s$. Here we assume that \mathcal{R} is a representation which does not break up into integer spin representations of $SU(2)$; otherwise we would be in trouble when $\gamma_0 = 0$. The fundamental representations of ADE groups except for E_8 are in accord with this assumption.

We now integrate out massive flavors to obtain low-energy $N = 1$ $SU(2)$ Yang-Mills theory with the dynamical scale Λ_{YM} . Reading off the flavor masses from (60) we get the scale matching

$$\begin{aligned} \Lambda_{YM}^{3/2} &= g_r^2 A(a), \\ A(a) &\equiv \Lambda^{2h-l(\mathcal{R})N_f} \prod_{s=1}^{n_{\mathcal{R}}} \left\{ \det \left(\sum_{l=0}^q \gamma_l (a \cdot \lambda_{\mathcal{R}_s})^l \right)^{l(\mathcal{R}_{SU(2)}^s)} \right\}, \end{aligned} \quad (61)$$

where $l(\mathcal{R}_{SU(2)}^s)$ is the index of $\mathcal{R}_{SU(2)}^s$ which is related to $l(\mathcal{R})$ through

$$l(\mathcal{R}) = \sum_{s=1}^{n_{\mathcal{R}}} l(\mathcal{R}_{SU(2)}^s). \quad (62)$$

The index of the spin $m/2$ representation of $SU(2)$ is given by $m(m+1)(m+2)/6$.

Including the effect of $SU(2)$ gaugino condensation we finally arrive at the effective superpotential for low-energy $SU(2)$ theory

$$W_L = W_{cl}(g) \pm 2\Lambda_{YM}^3 = W_{cl}(g) \pm 2g_r \sqrt{A(a)}, \quad (63)$$

The expectation values $\langle u_k \rangle = \partial W_L / \partial g_k$ are found to be

$$\begin{aligned}\langle u_j \rangle &= u_j^{cl} \pm 2 \frac{\partial \sqrt{A}}{\partial g'_j}, \quad 1 \leq j \leq r-1, \\ \langle u_r \rangle &= u_r^{cl} \pm 2 \left(\sqrt{A} + g_r \sum_{k=1}^{r-1} \frac{\partial g'_k}{\partial g_r} \frac{\partial \sqrt{A}}{\partial g'_k} \right) \\ &= u_r^{cl} \pm 2 \left(\sqrt{A} - \sum_{k=1}^{r-1} g'_k \frac{\partial \sqrt{A}}{\partial g'_k} \right),\end{aligned}\tag{64}$$

where we have set $g'_k = g_k/g_r$ and used the fact that u_k^{cl} and A are functions of g'_k since a_i in (63) are solutions of (4) (see also (26)).

Let us show that the vacuum expectation values (64) obey the singularity condition for the family of $(r-1)$ -dimensional complex manifolds defined by $\mathcal{W} = 0$ with coordinates z, x_1, \dots, x_{r-1} where

$$\mathcal{W} = z + \frac{A(x_n)}{z} - \sum_{i=1}^r x_i (u_i - u_i^{cl}(x_n)).\tag{65}$$

Here we have introduced the variables x_i ($1 \leq i \leq r-1$) instead of g'_i to express $A(g'_n)$ and $u_i^{cl}(g'_n)$, $x_r = 1$ and u_i are moduli parameters. The manifold $\mathcal{W} = 0$ is singular when

$$\frac{\partial \mathcal{W}}{\partial z} = 0, \quad \frac{\partial \mathcal{W}}{\partial x_i} = 0.\tag{66}$$

Then, if we set $z = \pm \sqrt{A(x_k)}$, $x_k = g'_k$ and $u_j = \langle u_j \rangle$ it is easy to show that the singularity conditions are satisfied

$$\begin{aligned}\mathcal{W} &= \pm 2\sqrt{A(g'_k)} - \sum_{i=1}^r g'_i (\langle u_i \rangle - u_i^{cl}(g'_k)) = 0, \\ \frac{\partial \mathcal{W}}{\partial z} &= 0, \\ \frac{\partial \mathcal{W}}{\partial x_j} &= \pm \frac{1}{\sqrt{A(g'_k)}} \frac{\partial A(g'_k)}{\partial g'_j} - \langle u_j \rangle + \frac{\partial}{\partial g'_j} \left(\sum_{i=1}^r g'_i u_i^{cl}(g'_k) \right) \\ &= -u_j^{cl}(g'_k) + g_r \frac{\partial}{\partial g_j} \left(\frac{W_{cl}(g)}{g_r} \right) = 0, \quad 1 \leq j \leq r-1.\end{aligned}\tag{67}$$

Thus the singularities of the manifold defined by $\mathcal{W} = 0$ are parametrized by the expectation values $\langle u_k \rangle$.

Let us explain how the known curves for $SU(N_c)$ and $SO(2N_c)$ supersymmetric QCD are reproduced from (65). First we consider $SU(N_c)$ theory with N_f fundamental flavors. Here we denote the degree i Casimir by u_i and correspondingly change the notations for x_j and g'_j . It is shown in [9],[10] that

$$A = \Lambda^{2N_c - N_f} \det_{N_f} \left(\sum_{l=0}^q (a^1)^l \gamma_l \right), \quad a^1 = g'_{N_c-1}, \quad (68)$$

and hence (65) becomes

$$\mathcal{W} = z + \frac{A(x_{N_c-1})}{z} - \sum_{i=2}^{N_c} x_i (u_i - u_i^{cl}(x_n)). \quad (69)$$

Since A depends only on x_{N_c-1} we notice that one can eliminate other variables x_1, \dots, x_{N_c-2} by imposing $\partial \mathcal{W} / \partial x_j = 0$ to get the relation

$$u_j^{cl}(x_n) = u_j \quad (70)$$

for $2 \leq j \leq N_c - 2$, and then

$$\mathcal{W} = z + \frac{A(x_{N_c-1})}{z} - (u_{N_c} - u_{N_c}^{cl}(x_n)) - x_{N_c-1} (u_{N_c-1} - u_{N_c-1}^{cl}(x_n)). \quad (71)$$

Remember that

$$0 = \det \left(a^1 - \Phi_{cl} \right) = (a^1)^{N_c} - s_2^{cl} (a^1)^{N_c-1} - \dots - s_{N_c}^{cl}, \quad (72)$$

where

$$ks_k + \sum_{i=1}^k is_{k-1} u_i = 0, \quad u_n = \frac{1}{n} \text{Tr } \Phi^n, \quad k = 1, 2, \dots \quad (73)$$

with $s_0 = -1$ and $s_1 = u_1 = 0$. We see with the aid of (72) that

$$\begin{aligned} u_{N_c}^{cl} + x_{N_c-1} u_{N_c-1}^{cl} &= (u_{N_c}^{cl} - s_{N_c}^{cl}) + x_{N_c-1} (u_{N_c-1}^{cl} - s_{N_c-1}^{cl}) + (s_{N_c}^{cl} + x_{N_c-1} s_{N_c-1}^{cl}) \\ &= (u_{N_c} - s_{N_c}) + x_{N_c-1} (u_{N_c-1} - s_{N_c-1}) \\ &\quad + \left((x_{N_c-1})^{N_c} - s_2 (x_{N_c-1})^{N_c-1} - \dots - s_{N_c-2} \right), \end{aligned} \quad (74)$$

where (70) and the fact that $s_{N_c} = u_{N_c} + (\text{polynomial of } u_k, 2 \leq k \leq N_c - 2)$ have been utilized. We now rewrite (71) as

$$\begin{aligned} \mathcal{W} &= z + \frac{A(x)}{z} - (u_{N_c} + xu_{N_c-1}) + (u_{N_c}^{cl} + xu_{N_c-1}^{cl}) \\ &= z + \frac{A(x)}{z} + x^{N_c} - s_2 x^{N_c-1} - \dots - s_{N_c}, \end{aligned} \quad (75)$$

where x_{N_c-1} was replaced by x for notational simplicity. This reproduces the hyperelliptic curve derived in [29],[10] after making a change of variable $y = z - A(x)/z$ and agrees with the $N = 2$ curve obtained in [30],[31],[32] in the $N = 2$ limit .

Next we consider $SO(2N_c)$ theory with $2N_f$ fundamental flavors Q . Following [10] we take a tree-level superpotential

$$W = \sum_{n=k}^{N_c-2} g_{2k} u_{2k} + g_{2(N_c-1)} s_{N_c-1} + \lambda v + \frac{1}{2} \sum_{l=0}^q \text{Tr}_{2N_f} \gamma_l Q \Phi^l Q, \quad (76)$$

where

$$\begin{aligned} u_{2k} &= \frac{1}{2k} \text{Tr} \Phi^{2k}, \quad 1 \leq k \leq N_c - 1, \\ v &= \text{Pf } \Phi = \frac{1}{2^{N_c} N_c!} \epsilon_{i_1 i_2 j_1 j_2 \dots} \Phi^{i_1 i_2} \Phi^{j_1 j_2} \dots \end{aligned} \quad (77)$$

and

$$ks_k + \sum_{i=1}^k i s_{k-i} u_{2i} = 0, \quad s_0 = -1, \quad k = 1, 2, \dots \quad (78)$$

According to [8] we have

$$(a^1)^2 = g'_{2(N_c-2)}, \quad \lambda' = 2 \prod_{j=2}^{N_c-1} (-ia^j), \quad v^{cl} = -g'_{2(N_c-2)} \lambda' / 2 \quad (79)$$

and [10]

$$A = \Lambda^{4(N_c-1)-2N_f} \det_{2N_f} \left(\sum_{l=0}^q (a^1)^l \gamma_l \right), \quad (80)$$

and thus

$$\mathcal{W} = z + \frac{A(x_{N_c-2})}{z} - \sum_{i=1}^{N_c-1} x_i (u_{2i} - u_{2i}^{cl}(x_n)) - x(v - v^{cl}(x_n)), \quad (81)$$

where $\lambda' = \lambda/g_{2(N_c-1)}$ was replaced by x and $g_{2i}/g_{2(N_c-1)}$ by x_i .

In view of (80) we again notice that there are redundant variables which can be eliminated by imposing the condition $\partial \mathcal{W} / \partial x_j = 0$ so as to obtain

$$u_{2j}^{cl}(x_n) = u_{2j} \quad (82)$$

for $1 \leq j \leq N_c - 3$. We then find

$$\begin{aligned} \mathcal{W} = z + \frac{A(x_{N_c-2})}{z} &- (u_{2(N_c-1)} - u_{2(N_c-1)}^{cl}(x_n)) - x_{N_c-2} (u_{2(N_c-2)} - u_{2(N_c-2)}^{cl}(x_n)) \\ &- x(v - v^{cl}(x_n)). \end{aligned} \quad (83)$$

Using $\det(a^1 - \Phi_{cl}) = 0$ we proceed further as in the $SU(N_c)$ case. The final result reads

$$\begin{aligned}\mathcal{W} &= z + \frac{A(y)}{z} + \frac{1}{y} \left(y^{N_c} - s_1 y^{N_c-1} - \cdots - s_{N_c-1} y + v^{cl}(x_n)^2 \right) \\ &\quad - x(v - v^{cl}(x_n)) \\ &= z + \frac{A(y)}{z} - \frac{1}{4} x^2 y + y^{N_c-1} - s_1 y^{N_c-2} - \cdots - s_{N_c-1} - vx,\end{aligned}\quad (84)$$

where we have set $y = x_{N_c-2}$ and used (79). It is now easy to check that imposing $\partial\mathcal{W}/\partial x = 0$ to eliminate x yields the known curve in [10] which has the correct $N = 2$ limit [33],[32].

It should be noted here that adding gaussian variables in (75) and (84) we have

$$\begin{aligned}\mathcal{W}_{A_{n-1}} &= z + \frac{A(y_1)}{z} + y_1^n - s_2 y_1^{n-1} - \cdots - s_n + y_2^2 + y_3^2, \\ \mathcal{W}_{D_n} &= z + \frac{A(y_1)}{z} - \frac{1}{4} y_2^2 y_1 + y_1^{n-1} - s_1 y_1^{n-2} - \cdots - s_{n-1} - v y_2 + y_3^2.\end{aligned}\quad (85)$$

These are equations describing ALE spaces of AD type fibered over \mathbf{CP}^1 . Inclusion of matter hypermultiplets makes fibrations more complicated than those for pure Yang-Mills theory. For A_n the result is rather obvious, but for D_n it may be interesting to follow how two variables y_1, y_2 come out naturally from (65). These variables are traced back to coupling constants $g_{2(n-2)}/g_{2(n-1)}$, $\lambda/g_{2(n-1)}$, respectively, and their degrees indeed agree $[y_1] = [g_{2(n-2)}/g_{2(n-1)}] = 2$, $[y_2] = [\lambda/g_{2(n-1)}] = n - 2$.

This observation suggests a possibility that even in the E_n case we may eliminate redundant variables and derive the desired ALE form of SW geometry directly from (65) although the problem certainly becomes non-linear. The issue is under current investigation.

5 Conclusions

We have obtained a low-energy effective superpotential for a phase with a single confined photon in $N = 1$ gauge theory with an adjoint matter with ADE gauge groups. The expectation values of gauge invariants built out of the adjoint field parametrize the singularities of moduli space of the $N = 2$ Coulomb phase. The result can be used to derive the $N = 2$ curve in the form of a foliation over \mathbf{CP}^1 . According to our derivation it

is clearly observed that the quantum effect in the SW curve has its origin in the $SU(2)$ gluino condensation in view of $N = 1$ gauge theory dynamics.

In the last year it has been clarified how SW geometry of $N = 2$ Yang-Mills theory appears naturally in the context of type II compactification on Calabi-Yau threefolds [34],[14],[1]. For gauge groups of ADE type SW geometry is derived from ALE spaces of type ADE fibered over \mathbf{CP}^1 . Furthermore SW curves obtained from ALE fibrations take the form of spectral curves for the ADE periodic Toda lattice. These Toda spectral curves are described as foliations over \mathbf{CP}^1 . Thus it seems that ALE fibrations combined with integrable systems provide us with a natural point of view for SW geometry and the SW curve in the form of a foliation over \mathbf{CP}^1 is recognized as a canonical description. Our study of ADE confining phase superpotentials also supports this point of view. It is highly desirable to develop such a scheme explicitly for non-simply-laced gauge groups and supersymmetric gauge theories with fundamental matter fields.

The work of ST is supported by JSPS Research Fellowship for Young Scientists. The work of SKY was supported in part by the Grant-in-Aid for Scientific Research on Priority Area 213 “Infinite Analysis”, the Ministry of Education, Science and Culture, Japan. SKY would like to thank the ITP in Santa Barbara for hospitality where a part of this work was done. This work was supported in part by the National Science Foundation under Grant No. PHY94-07194.

References

- [1] For recent reviews, see, W. Lerche, *Introduction to Seiberg-Witten Geometry and its Stringy Origin*, hep-th/9611190; C. Vafa, *Lectures on Strings and Dualities*, hep-th/9702201; A. Klemm, *On the Geometry behind $N=2$ Supersymmetric Effective Actions in Four Dimensions*, hep-th/9705131
- [2] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19; Nucl. Phys. **B431** (1994) 484
- [3] K. Intriligator and N. Seiberg, Nucl. Phys. **B431** (1994) 551
- [4] S. Elitzur, A. Forge, A. Giveon and E. Rabinovici, Phys. Lett. **B353** (1995) 79
- [5] S. Elitzur, A. Forge, A. Giveon and E. Rabinovici, Nucl. Phys. **B459** (1996) 160
- [6] K. Intriligator, Phys. Lett. **B336** (1994) 409
- [7] S. Elitzur, A. Forge, A. Giveon, K. Intriligator and E. Rabinovici, Phys. Lett. **B379** (1996) 121
- [8] S. Terashima and S.-K. Yang, Phys. Lett. **B391** (1997) 107
- [9] T. Kitao, *SUSY $N=2$ hyperelliptic curve from $N=1$ effective potential*, hep-th/9611097
- [10] T. Kitao, S. Terashima and S.-K. Yang, Phys. Lett. **B399** (1997) 75
- [11] A. Giveon, O. Pelc and E. Rabinovici, *The Coulomb Phase in $N=1$ Gauge Theories With a LG-Type Superpotential*, hep-th/9701045
- [12] K. Landsteiner, J.M. Pierre and S.B. Giddings, Phys. Rev. **D55** (1997) 2367
- [13] E.J. Martinec and N.P. Warner, Nucl. Phys. **B459** (1996) 97
- [14] W. Lerche and N.P. Warner, *Exceptional SW Geometry from ALE Fibrations*, hep-th/9608183

- [15] N.P. Warner and S.-K. Yang, to appear
- [16] K. Ito, *Picard-Fuchs Equations and Prepotential in $N=2$ Supersymmetric G_2 Yang-Mills Theory*, hep-th/9703180, to appear in *Phys. Lett. B*
- [17] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press (1990)
- [18] K. Intriligator and N. Seiberg, *Nucl. Phys. B* **444** (1995) 125.
- [19] A. Gorskii, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, *Phys. Lett. B* **355** (1995) 466
- [20] T. Nakatsu and K. Takasaki, *Mod. Phys. Lett. A* **11** (1996) 157
- [21] H. Itoyama and A. Morozov, *Nucl. Phys. B* **477** (1996) 855; *Nucl. Phys. B* **491** (1997) 529
- [22] A. Brandhuber and K. Landsteiner, *Phys. Lett. B* **358** (1995) 73
- [23] P.C. Argyres and M.R. Douglas, *Nucl. Phys. B* **448** (1995) 93
- [24] P.C. Argyres, M.R. Plesser, N. Seiberg and E. Witten, *Nucl. Phys. B* **461** (1996) 71
- [25] T. Eguchi, K. Hori, K. Ito and S.-K. Yang, *Nucl. Phys. B* **471** (1996) 430
- [26] T. Eguchi and K. Hori, *$N=2$ Superconformal Field Theories in 4 Dimensions and A-D-E Classification*, hep-th/9607125
- [27] D. Kutasov, A. Schwimmer and N. Seiberg, *Nucl. Phys. B* **459** (1996) 455
- [28] R. Slansky, *Phys. Reports* **79** (1981) 1
- [29] A. Kapustin, *Phys. Lett. B* **398** (1997) 104
- [30] A. Hanany and Y. Oz, *Nucl. Phys. B* **452** (1995) 283
- [31] P.C. Argyres, M.R. Plesser and A.D. Shapere, *Phys. Rev. Lett.* **75** (1995) 1699
- [32] P.C. Argyres and A.D. Shapere, *Nucl. Phys. B* **461** (1996) 437

[33] A. Hanany, Nucl. Phys. **B466** (1996) 85

[34] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, Nucl. Phys. **B477** (1996) 746